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# Cone orderings, group majorizations and similarly separable vectors

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ABSTRACT

We generalize some results on majorization in papers by Wu and Debnath [S. Wu, L. Debnath, Inequalities for convex sequences and their applications, Comput. Math. Appl. 54 (2007) 525–534] and by Marshall et al. [A.W. Marshall, I. Olkin, F. Proschan, Monotonicity of ratios of means and other applications of majorization, in: O. Shisha (Ed.), Inequalities, Academic Press, New York, 1967, pp. 177–190]. We present sufficient conditions for some vector inequalities to hold in the case of cone orderings and group-induced cone orderings. The framework used is based on results for similarly separable vectors given in paper [M. Niezgoda, Bifractional inequalities and convex cones, Discrete Math. 306 (2006) 231–243].


## 1. Motivation

By $z_{[1]} \geqslant z_{[2]} \geqslant \cdots \geqslant z_{[n]}$ we denote the entries of a vector $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ arranged in decreasing order.

A vector $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is said to be *majorized* by vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ (in symbols, $y \prec_m x$) if $\sum_{k=1}^{i} y_{[k]} \leqslant \sum_{k=1}^{i} x_{[k]}$ for all $i = 1, 2, \ldots, n$ with equality for $i = n$ (see [5, p. 7]).

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It is well-known by Rado's theorem that for any $x, y \in \mathbb{R}^n$,

$$y \prec_m x \quad \text{iff } y \in \operatorname{conv} \mathbb{P}_n x, \tag{1}$$

where $\mathbb{P}_n$ is the group of all $n \times n$ permutation matrices, and $\operatorname{conv} \mathbb{P}_n x$ is the convex hull of the set $\mathbb{P}_n x = \{xp : p \in \mathbb{P}_n\}$ (see [14]).

Wu and Debnath [15, Lemma 4] (cf. Marshall and Olkin [5, Theorem B.1, p. 129]) have shown, among other results, the following.

**Theorem A** [15]. *Let $x, y \in \mathbb{R}^n$, $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$, $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If there exists $m$ $(1 \leqslant m \leqslant n$, $m \in \mathbb{N})$ such that*

$$y_i \leqslant x_i \text{ for } i = 1, 2, \ldots, m \quad \text{and} \quad y_j \geqslant x_j \text{ for } j = m+1, m+2, \ldots, n,$$

*then*

$$y \prec_m x. \tag{2}$$

By a result of Marshall et al. [4, Theorem 2.4] (see also [3, Lemma 2.1], cf. Marshall and Olkin [5, Theorem B.1.b, p. 129]) it is known that

**Theorem B** [4]. *If $x_i > 0$, $1 \leqslant i \leqslant n$, and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n > 0$ and*

$$\frac{x_1}{y_1} \geqslant \cdots \geqslant \frac{x_n}{y_n},$$

*then*

$$\frac{y}{\sum_{j=1}^n y_j} \prec_m \frac{x}{\sum_{j=1}^n x_j}, \tag{3}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$.

See [13, Theorem 3] for a continuous version of Theorem B.

Let $V$ be a finite-dimensional real linear space with inner product $\langle \cdot, \cdot \rangle$, and let $G$ be a closed subgroup of the orthogonal group $O(V)$ acting on $V$.

A vector $y \in V$ is said to be *$G$-majorized* by vector $x \in V$, denoted $y \prec_G x$, if $y$ is an element of the convex hull $\operatorname{conv} Gx$ of the $G$-orbit $Gx = \{gx : g \in G\}$ [1]. That is,

$$y \prec_G x \quad \text{iff } y \in \operatorname{conv} Gx \tag{4}$$

(cf. (1)). The ordering $\prec_G$ on $V$ is called the *group majorization* w.r.t. $G$, abbreviated as *$G$-majorization*.

In this paper we extend Theorems A and B from the classical majorization $\prec_m$ to a class of $G$-majorizations $\prec_G$ called *group-induced cone orderings* [12]. We give some sufficient conditions for inequalities of type (3) and (2) to hold in the case of cone orderings (see Section 2) and group-induced cone orderings with finite $G$ (see Section 3). To do so, we use the notion of the *similar separability* of two vectors (see [10,9]). In Sections 4–6 we interpret our results for weak absolute majorization, classical majorization and Miranda–Thompson's majorization, respectively.

## 2. Results for cone orderings

Let $V$ be a finite-dimensional real linear space with inner product $\langle \cdot, \cdot \rangle$. A nonempty set $C \subset V$ is said to be a *convex cone* if $\alpha C + \beta C \subset C$ for all $0 \leqslant \alpha, \beta \in \mathbb{R}$.

We define a *cone ordering* $\leqslant_C$ on $V$ induced by a convex cone $C \subset V$ as follows. For $x, y \in V$,

$$y \leqslant_C x \quad \text{iff } x - y \in C.$$

The *dual cone* of a convex cone $C \subset V$ is the convex cone in $V$ defined by

$$\operatorname{dual} C = \{w \in V : \langle w, u\rangle \geqslant 0 \ \text{ for all } u \in C\}.$$

For a given (nonempty) subset $A \subset V$, we denote by $\operatorname{cone} A$ the convex cone of all nonnegative finite combinations of vectors in $A$.

Let $e = (e_1, \ldots, e_n)$ be an ordered basis in $V$ (with $\dim V = n$). Let $I_1$ and $I_2$ be two sets of indices such that $I_1 \cup I_2 = I$, where $I = \{1, 2, \ldots, n\}$. For given $\mu \in \mathbb{R}$ and $v \in V$, we say that a vector $z \in V$ is $\mu, v$*-separable* on $I_1$ and $I_2$ (with respect to the basis $e$) if

$$\langle e_i, z - \mu v\rangle \geqslant 0 \quad \text{for } i \in I_1 \quad \text{and} \quad \langle e_j, z - \mu v\rangle \leqslant 0 \quad \text{for } j \in I_2, \tag{5}$$

or equivalently, if

$$\frac{\langle e_i, z\rangle}{\langle e_i, v\rangle} \geqslant \mu \geqslant \frac{\langle e_j, z\rangle}{\langle e_j, v\rangle} \quad \text{for } i \in I_1 \text{ and } j \in I_2$$

(provided the denominators are positive). See [9–11] for details.

A vector $z \in V$ is said to be $v$*-separable* on $I_1$ and $I_2$ (w.r.t. $e$) if $z$ is $\mu, v$-separable on $I_1$ and $I_2$ (w.r.t. $e$) for some $\mu$.

We say that two (ordered) bases $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ in $V$ are *dual* if $\langle e_i, d_j\rangle = \delta_{ij}$ (Kronecker delta), $i, j = 1, 2, \ldots, n$.

Let $x, y, z, v \in V$ and $\lambda, \mu \in \mathbb{R}$. The vectors $x, z \in V$ are said to be *similarly separable* w.r.t. $(\lambda, y, e; \mu, v, d)$ if there exist two index sets $I_1$ and $I_2$ with $I_1 \cup I_2 = \{1, 2, \ldots, n\}$ such that

**(i)** $x$ is $\lambda, y$-separable on $I_1$ and $I_2$ w.r.t. $e$,
**(ii)** $z$ is $\mu, v$-separable on $I_1$ and $I_2$ w.r.t. $d$

(see [11]).

We begin our discussion of generalizations of Theorems A and B with a preliminary result which is a reformulation of [9, Theorem 3.5].

**Theorem 2.1** (See [9, Theorem 3.5]). *Let $x, y, v \in V$ and $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$ with $s_1, s_2, \ldots, s_n \in V$.*

*Suppose that for each $k = 1, 2, \ldots, n$ there exist dual bases $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ of $V$ and index sets $I_1$ and $I_2$ with $I_1 \cup I_2 = \{1, 2, \ldots, n\}$ such that any of the following conditions* **(a)**–**(d)** *is satisfied with $z = s_k$.*

**(a)** *$x$ is $\lambda_1, y$-separable on $I_1$ and $I_2$ w.r.t. $e$ with $\lambda_1 = \langle x, v\rangle \,/\, \langle y, v\rangle$ and $\langle y, v\rangle > 0$, and $z$ is $v$-separable on $I_1$ and $I_2$ w.r.t. $d$.*
**(b)** *$y$ is $\lambda_2, x$-separable on $I_1$ and $I_2$ w.r.t. $e$ with $\lambda_2 = \langle y, z\rangle \,/\, \langle x, z\rangle$ and $\langle x, z\rangle > 0$, and $v$ is $z$-separable on $I_1$ and $I_2$ w.r.t. $d$.*
**(c)** *$z$ is $\lambda_3, v$-separable on $I_1$ and $I_2$ w.r.t. $e$ with $\lambda_3 = \langle z, y\rangle \,/\, \langle v, y\rangle$ and $\langle v, y\rangle > 0$, and $x$ is $y$-separable on $I_1$ and $I_2$ w.r.t. $d$.*
**(d)** *$v$ is $\lambda_4, z$-separable on $I_1$ and $I_2$ w.r.t. $e$ with $\lambda_4 = \langle v, x\rangle \,/\, \langle z, x\rangle$ and $\langle z, x\rangle > 0$, and $y$ is $x$-separable on $I_1$ and $I_2$ w.r.t. $d$.*

*Then the following cone inequality holds:*

$$\langle x, v\rangle\, y \leqslant_{\operatorname{dual} D} \langle y, v\rangle\, x. \tag{6}$$

*If in addition $\langle x, v\rangle\ \langle y, v\rangle > 0$ then (6) can be restated as*

$$\frac{y}{\langle y, v\rangle} \leqslant_{\operatorname{dual} D} \frac{x}{\langle x, v\rangle}. \tag{7}$$

**Proof.** Denote

$$T(a, b, u, w) = \langle a, b\rangle \langle u, w\rangle - \langle a, w\rangle \langle b, u\rangle \quad \text{for } a, b, u, w \in V.$$

Then one has

$$\begin{aligned}\langle \langle y, v\rangle x - \langle x, v\rangle y, z\rangle &= T(x, z, y, v) = T(y, v, x, z)\\ &= T(z, x, v, y) = T(v, y, z, x).\end{aligned} \tag{8}$$

In the case of assumption **(a)**, we obtain $T(x, z, y, v) \geqslant 0$ by using [9, Theorem 3.5]. Consequently, we get the following inequality:

$$\langle \langle y, v\rangle x - \langle x, v\rangle y, z\rangle \geqslant 0 \quad \text{with } z = s_k \text{ for } k = 1, 2, \ldots, n, \tag{9}$$

which amounts to (6), as required.

Under each of the assumptions **(b)**, **(c)** or **(d)**, inequalities (9) and (6) follow from [9, Theorem 3.5] and identities (8).

Finally, inequality (7) is a simple reformulation of (6). □

**Remark 2.2.** The key property of assumption **(a)** of Theorem 2.1 is that the vectors $x$ and $z = s_k$ are *similarly* separable w.r.t. $(\lambda_1, y, e; \mu, v, d)$ for some $\mu \in \mathbb{R}$. Assumptions **(b)**, **(c)** and **(d)** can be interpreted in a similar fashion.

We now provide some sufficient conditions for cone inequalities (13) and (14) to hold.

**Theorem 2.3.** *Let $x, y, v \in V$ be such that $\langle x, v\rangle > 0$ and $\langle y, v\rangle > 0$, and let $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$ with $s_1, s_2, \ldots, s_n \in V$.*

*Suppose that there exist a basis $e = (e_1, e_2, \ldots, e_n)$ of $V$ and an index $m \in \{0, 1, 2, \ldots, n\}$ such that*

***(i)***

$$x \text{ is } \lambda, y\text{-separable on } I_1 \text{ and } I_2 \text{ w.r.t. } e, \tag{10}$$

*where $\lambda = \langle x, v\rangle / \langle y, v\rangle$ and $I_1 = \{1, 2, \ldots, m\}$, $I_2 = \{m + 1, m + 2, \ldots, n\}$,*

***(ii)***

$$0 \leqslant_{C_1} s_i \quad \text{and} \quad s_j \leqslant_{C_2} v \quad \text{for } i \in I_1 \text{ and } j \in I_2, \tag{11}$$

*where*

$$C_1 = \operatorname{cone}\{e_1, e_2, \ldots, e_m\} \quad \text{and} \quad C_2 = \operatorname{cone}\{e_{m+1}, e_{m+2}, \ldots, e_n\}. \tag{12}$$

*Then the following inequality holds:*

$$\frac{y}{\langle y, v\rangle} \leqslant_{\operatorname{dual} D} \frac{x}{\langle x, v\rangle}. \tag{13}$$

*If in addition $\langle x, v\rangle = \langle y, v\rangle$ then*

$$y \leqslant_{\operatorname{dual} D} x. \tag{14}$$

**First proof** (*Based on the idea of the proof of* [15, Lemma 4]).

By (10) we get

$$\langle x - \lambda y, e_i\rangle \geqslant 0 \quad \text{for } i \in I_1 \tag{15}$$

and

$$\langle x - \lambda y, e_j \rangle \leqslant 0 \quad \text{for } j \in I_2. \tag{16}$$

On account of (11) and (12) we have

$$s_i = \alpha_{i1} e_1 + \cdots + \alpha_{im} e_m \quad \text{for } i \in I_1 \tag{17}$$

and

$$v - s_j = \beta_{j,m+1} e_{m+1} + \cdots + \beta_{j,n} e_n \quad \text{for } j \in I_2, \tag{18}$$

for some $\alpha_{i,k} \geqslant 0\,(i, k = 1, \ldots, m)$ and $\beta_{j,l} \geqslant 0\,(j, l = m+1, \ldots, n)$, respectively.

It follows from (15) and (17) that

$$\langle x - \lambda y, s_i \rangle \geqslant 0 \quad \text{for } i \in I_1. \tag{19}$$

We shall show that

$$\langle x - \lambda y, s_j \rangle \geqslant 0 \quad \text{for } j \in I_2. \tag{20}$$

By (18), we obtain

$$s_j = v - \sum_{l=m+1}^{n} \beta_{jl} e_l \quad \text{for } j \in I_2. \tag{21}$$

On the other hand, a simple calculation reveals that

$$\langle x - \lambda y, v \rangle = 0. \tag{22}$$

By (21), (22) and (16), for $j \in I_2$ we derive

$$\langle x - \lambda y, s_j \rangle = \left\langle x - \lambda y, v - \sum_{l=m+1}^{n} \beta_{jl} e_l \right\rangle = \langle x - \lambda y, v \rangle - \sum_{l=m+1}^{n} \beta_{jl} \langle x - \lambda y, e_l \rangle \geqslant 0.$$

This completes the proof of (20).

By combining (19) and (20), we get (13). □

**Second proof** (*Based on our Theorem* 2.1).

Let $d = (d_1, \ldots, d_n)$ be the dual basis of $e = (e_1, \ldots, e_n)$ in $V$. We shall show that for each $k = 1, 2, \ldots, n$, condition **(a)** of Theorem 2.1 is fulfilled for $z = s_k$.

Observe that assumption (10) is the first part of **(a)**.

To prove the second part of **(a)**, fix arbitrarily $k \in \{1, 2, \ldots, n\}$.

If $1 \leqslant k \leqslant m$ then by making use of (11)–(12) we establish

$$\langle s_k, d_i \rangle \geqslant 0 \quad \text{and} \quad \langle s_k, d_j \rangle = 0 \quad \text{for } i \in I_1 \text{ and } j \in I_2.$$

Hence

$$\langle s_k, d_i \rangle \geqslant 0 \, \langle v, d_i \rangle \quad \text{and} \quad \langle s_k, d_j \rangle \leqslant 0 \, \langle v, d_j \rangle \quad \text{for } i \in I_1 \text{ and } j \in I_2.$$

In other words, $s_k$ is $0$, $v$-separable on $I_1$ and $I_2$ w.r.t. $d$ (see (5)).

Likewise, if $m + 1 \leqslant k \leqslant n$ then from (11)-(12) we find that

$$\langle v - s_k, d_i \rangle = 0 \quad \text{and} \quad \langle v - s_k, d_j \rangle \geqslant 0 \quad \text{for } i \in I_1 \text{ and } j \in I_2.$$

This gives

$$\langle s_k, d_i \rangle \geqslant 1 \langle v, d_i \rangle \quad \text{and} \quad \langle s_k, d_j \rangle \leqslant 1 \langle v, d_j \rangle \quad \text{for } i \in I_1 \text{ and } j \in I_2.$$

That is, $s_k$ is 1, $v$-separable on $I_1$ and $I_2$ w.r.t. $d$ (see (5)).

The result now follows from Theorem 2.1. □

**Remark 2.4.** As proved above, if conditions (11) and (12) are satisfied then for each $k = 1, 2, \ldots, n$, the vector $s_k$ is $v$-separable on $I_1 = \{1, 2, \ldots, m\}$ and $I_2 = \{m+1, m+2, \ldots, n\}$ w.r.t. the dual basis $d$ of $e$.

To illustrate conditions (11) and (12) in Theorem 2.3, we need the following direct result.

**Lemma 2.5.** *Let $M = (c_{i,j})$ be an $n \times n$ real matrix and let $e_1, \ldots, e_n$ and $s_1, \ldots, s_n$ be vectors in $V$ such that*

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \ldots & c_{1n} \\ c_{21} & c_{22} & \ldots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \ldots & c_{nn} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} \tag{23}$$

*in the sense that*

$$s_i = \sum_{j=1}^{n} c_{ij} e_j \quad \textit{for } i = 1, 2, \ldots, n. \tag{24}$$

*Assume that $M$ has the block form*

$$M = \begin{pmatrix} A & 0 \\ K & B \end{pmatrix}, \tag{25}$$

*where $A = (\alpha_{ik})$ $(i, k = 1, 2, \ldots, m)$ is an $m \times m$ matrix with*

$$\alpha_{ik} \geqslant 0 \quad \textit{for } i, k = 1, 2, \ldots, m, \tag{26}$$

*and $B = (\beta_{jl})$ $(j, l = m+1, m+2, \ldots, n)$ is an $n - m \times n - m$ matrix, and*

$$K = \begin{pmatrix} \gamma_1 & \gamma_2 & \ldots & \gamma_m \\ \gamma_1 & \gamma_2 & \ldots & \gamma_m \\ \vdots & \vdots & & \vdots \\ \gamma_1 & \gamma_2 & \ldots & \gamma_m \end{pmatrix} \quad \textit{for some } \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}. \tag{27}$$

*Furthermore, suppose that $v$ is a vector in $V$ such that*

$$v = \gamma_1 e_1 + \gamma_2 e_2 + \cdots + \gamma_m e_m + \delta_{m+1} e_{m+1} + \cdots + \delta_n e_n \tag{28}$$

*and*

$$\delta_l \geqslant \beta_{jl} \quad \text{for } j, l = m+1, m+2, \ldots, n. \tag{29}$$

*Then conditions (11) and (12) of Theorem 2.3 are satisfied.*

**Proof.** Straightforward computations. □

In the sequel, for a given vector $v \in V$ and a basis $e = (e_1, e_2, \dots, e_n)$ of $V$, and index sets $I_1, I_2$, we denote

$$S_e(\lambda, v; I_1, I_2) = \{z \in V : z \text{ is } \lambda, v\text{-separable on } I_1 \text{ and } I_2 \text{ w.r.t. } e\},$$

$$S_e(v; I_1, I_2) = \{z \in V : z \text{ is } v\text{-separable on } I_1 \text{ and } I_2 \text{ w.r.t. } e\}.$$

**Lemma 2.6.** *Let $x, y, z, v \in V$ and let $e = (e_1, e_2, \dots, e_n)$ and $d = (d_1, d_2, \dots, d_n)$ be dual bases in $V$. Let $\mathcal{I}_1 \times \mathcal{I}_2$ be a family of pairs $(I_1, I_2)$ of index sets such that $I_1 \cup I_2 = \{1, 2, \dots, n\}$. Assume that*

***(i)*** *there exists $(I_1, I_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ such that $x$ is $\lambda, y$-separable on $I_1$ and $I_2$ w.r.t. e, i.e.,*

$$x \in \bigcup_{(I_1, I_2) \in \mathcal{I}_1 \times \mathcal{I}_2} S_e(\lambda, y; I_1, I_2),$$

***(ii)*** *for all $(I_1, I_2) \in \mathcal{I}_1 \times \mathcal{I}_2$, $z$ is $v$-separable w.r.t. d, i.e.,*

$$z \in \bigcap_{(I_1, I_2) \in \mathcal{I}_1 \times \mathcal{I}_2} S_d(v; I_1, I_2).$$

*Then the vectors $x$ and $z$ are similarly separable w.r.t. $(\lambda, y, e; \mu, v, d)$ for some $\mu \in \mathbb{R}$.*

**Proof.** Evident. □

**Remark 2.7.** It is not hard to verify that Theorem 2.1 and Theorem 2.3 remain valid if conditions **(i)**–**(ii)** of Lemma 2.6 are used instead of condition **(a)** of Theorem 2.1 and of conditions **(i)**–**(ii)** of Theorem 2.3, respectively.

For example, assume that

$$\frac{\langle z, d_1 \rangle}{\langle v, d_1 \rangle} \geqslant \frac{\langle z, d_2 \rangle}{\langle v, d_2 \rangle} \geqslant \cdots \geqslant \frac{\langle z, d_n \rangle}{\langle v, d_n \rangle} \quad \text{(with positive denominators).}$$

Then condition **(ii)** of Lemma 2.6 is met for

$$\mathcal{I}_1 = \{I_{1,m} = \{1, 2, \dots, m\} : m = 0, 1, 2, \dots, n\}$$

and

$$\mathcal{I}_2 = \{I_{2,m} = \{m + 1, m + 2, \dots, n\} : m = 0, 1, 2, \dots, n\}$$

with notation $I_{1,0} = \emptyset$ and $I_{2,n} = \emptyset$.

Similarly, condition **(i)** of Lemma 2.6 holds if

$$\frac{\langle x, e_1 \rangle}{\langle y, e_1 \rangle} \geqslant \frac{\langle x, e_2 \rangle}{\langle y, e_2 \rangle} \geqslant \cdots \geqslant \frac{\langle x, e_n \rangle}{\langle y, e_n \rangle} \quad \text{(with positive denominators).}$$

## 3. Results for group-induced cone orderings

Throughout this section, unless stated otherwise, $V$ is a finite-dimensional real linear space equipped with inner product $\langle \cdot, \cdot \rangle$, and $G$ is a closed subgroup of the orthogonal group $O(V)$ acting on $V$. In addition, $\prec_G$ is the group majorization on $V$ defined by (4).

We say that $\prec_G$ is a *group-induced cone (GIC) ordering* if there exists a nonempty closed convex cone $F \subset V$ such that

**(A1)** $F \cap Gx$ is not empty for each $x \in V$,
**(A2)** $\langle x, gy \rangle \leqslant \langle x, y \rangle$ for all $x, y \in F$ and $g \in G$

(see [2]).

For examples of GIC orderings and groups satisfying axioms **(A1)**–**(A2)**, see [1,2,6–8].

If axioms **(A1)** and **(A2)** hold, then for any $x \in V$, the set $F \cap Gx$ has only one element denoted by $x_\downarrow$ [8, p. 14]. It is clear that for $x \in V$ we have

$$x_\downarrow = x \quad \text{iff } x \in F. \tag{30}$$

Any GIC ordering $\prec_G$, restricted to its cone $F$, is the cone ordering induced by $C = dualF$. In fact, for $x, y \in V$, the following inequalities are equivalent:

$$y \prec_G x, \tag{31}$$

$$y_\downarrow \prec_G x_\downarrow, \tag{32}$$

$$\langle y_\downarrow, z \rangle \leqslant \langle x_\downarrow, z \rangle \quad \text{for } z \in F, \tag{33}$$

$$\langle y_\downarrow, s_i \rangle \leqslant \langle x_\downarrow, s_i \rangle \quad \text{for } i \in J, \text{ provided } F = \operatorname{cone}\{s_i : i \in J\} \tag{34}$$

(see [1, p. 15] [8, pp. 13–14] and references therein).

The next result is a $G$-majorization complement to Theorem 2.1.

**Theorem 3.1.** *Assume that $\prec_G$ is a GIC ordering on $V$ induced by finite group $G$ and convex cone $F = \operatorname{cone}\{s_1, s_2, \ldots, s_q\}$ with $s_1, s_2, \ldots, s_q \in V$. Set $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$, where $s_1, s_2, \ldots, s_n$ form a basis of $V$, $n \leqslant q$.*

*Let $x, y, v \in V$ with $\langle x, v \rangle\, y \in F$ (e.g., $\langle x, v \rangle \geqslant 0$ and $y \in F$).*

*Suppose that for $k = 1, 2, \ldots, n$ there exist dual bases $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ of $V$, and index sets $I_1$ and $I_2$ with $I_1 \cup I_2 = \{1, 2, \ldots, n\}$ such that*

***(i)*** *$x$ is $\lambda, y$-separable on $I_1$ and $I_2$ w.r.t. $e$ with $\lambda = \langle x, v \rangle \,/\, \langle y, v \rangle$ and $\langle y, v \rangle > 0$,*
***(ii)*** *$s_k$ is $v$-separable on $I_1$ and $I_2$ w.r.t. $d$.*

*If $n < q$, assume additionally that*

$$\langle \langle x, v \rangle\, y, s_k \rangle \leqslant \langle \langle y, v \rangle\, x, s_k \rangle \quad \textit{for } \; k = n + 1, \ldots, q. \tag{35}$$

*Then the following $G$-majorization inequality holds:*

$$\langle x, v \rangle\, y \prec_G \langle y, v \rangle\, x. \tag{36}$$

*If $\langle x, v \rangle \; \langle y, v \rangle > 0$ then (36) becomes*

$$\frac{y}{\langle y, v \rangle} \prec_G \frac{x}{\langle x, v \rangle}.$$

**Remark 3.2.** In Theorem 3.1, if $q = n$ then the assumption (35) is *absent* (see Sections 4 and 6).

**Remark 3.3.** In Theorem 3.1, if $q = n + 1$ and $v = s_n = -s_{n+1}$ then the assumption (35) is fulfilled automatically and therefore can be dropped off (see Section 5).

**Proof of Theorem 3.1.** Observe that the above statements **(i)**–**(ii)** constitute condition **(a)** in Theorem 2.1 with $z = s_k$, $k = 1, 2, \ldots, n$. By virtue of Theorem 2.1 we obtain the inequality

$$\langle x, v\rangle\, y \leqslant_{\mathrm{dual}\, D} \langle y, v\rangle\, x. \tag{37}$$

That is,

$$\langle \langle x, v\rangle\, y, s_k\rangle \leqslant \langle \langle y, v\rangle\, x, s_k\rangle \quad \text{for } k = 1, 2, \ldots, n. \tag{38}$$

By (38) and (35) we get

$$\langle \langle x, v\rangle\, y, s_k\rangle \leqslant \langle \langle y, v\rangle\, x, s_k\rangle \quad \text{for } k = 1, 2, \ldots, n, n+1, \ldots, q.$$

Hence

$$\langle \langle x, v\rangle\, y, z\rangle \leqslant \langle \langle y, v\rangle\, x, z\rangle \quad \text{for } z \in F. \tag{39}$$

Simultaneously, we have

$$\langle \langle y, v\rangle\, x, z\rangle \leqslant \big\langle (\langle y, v\rangle\, x)_{\downarrow}, z\big\rangle \quad \text{for } z \in F \tag{40}$$

by axiom **(A2)** in the definition of a GIC ordering.

Also, as $\langle x, v\rangle\, y \in F$, we find from (30) that

$$(\langle x, v\rangle\, y)_{\downarrow} = \langle x, v\rangle\, y. \tag{41}$$

Combining (39), (40) and (41) yields

$$\big\langle (\langle x, v\rangle\, y)_{\downarrow}, z\big\rangle \leqslant \big\langle (\langle y, v\rangle\, x)_{\downarrow}, z\big\rangle \quad \text{for } z \in F.$$

So, by (31)–(34) we derive (36), as required. □

We are now in a position to present a $G$-majorization result corresponding to Theorem 2.3. Theorem 3.4 extends the mentioned result of Wu and Debnath [15] (see Theorem A in Section 1) from the classical majorization $\prec_m$ on $\mathbb{R}^n$ to a group-induced cone ordering $\prec_G$ with finite $G$.

**Theorem 3.4.** *Assume that $\prec_G$ is a GIC ordering on $V$ induced by finite group $G$ and convex cone $F = \operatorname{cone}\{s_1, s_2, \ldots, s_q\}$ with $s_1, s_2, \ldots, s_q \in V$. Set $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$, where $s_1, s_2, \ldots, s_n$ form a basis of $V$, $n \leqslant q$.*

*Let $x, y, v \in V$ be such that $\langle x, v\rangle > 0$, $\langle y, v\rangle > 0$ and $y \in F$.*

*Suppose that there exist a basis $e = (e_1, e_2, \ldots, e_n)$ and an index $m \in \{0, 1, 2, \ldots, n\}$ such that*

- **(i)** *$x$ is $\lambda, y$-separable on $I_1$ and $I_2$ w.r.t. $e$, where $\lambda = \langle x, v\rangle / \langle y, v\rangle$ and $I_1 = \{1, 2, \ldots, m\}$, $I_2 = \{m+1, m+2, \ldots, n\}$,*
- **(ii)** *vectors $s_1, s_2, \ldots, s_n, e_1, e_2, \ldots, e_n$ and $v$ satisfy conditions (11)–(12), or (23)–(29) for some $n \times n$ matrix $M = (c_{ij})$.*

*If $n < q$, assume additionally that (35) holds.*

*Then the following inequality holds:*

$$\frac{y}{\langle y, v\rangle} \prec_G \frac{x}{\langle x, v\rangle}. \tag{42}$$

*If in addition $\langle x, v\rangle = \langle y, v\rangle$ then*

$$y \prec_G x.$$

**Proof.** Use Theorem 2.3 to obtain (13) and (37). Next, apply the same method as in the proof of Theorem 3.1. □

In Corollary 3.5 we generalize Theorem B due to Marshall et al. [4] (see Section 1 for details).

**Corollary 3.5.** *Assume that $\prec_G$ is a GIC ordering on $V$ induced by finite group $G$ and convex cone $F = \operatorname{cone}\{s_1, s_2, \ldots, s_q\}$ with $s_1, s_2, \ldots, s_q \in V$. Set $D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\}$, where $s_1, s_2, \ldots, s_n$ form a basis of $V$, $n \leqslant q$.*

*Let $x, y, v \in V$ be such that $\langle x, v\rangle > 0$, $\langle y, v\rangle > 0$ and $y \in F$.*

*Suppose that there exists a basis $e = (e_1, e_2, \ldots, e_n)$ of $V$ such that*

***(i)***

$$\frac{\langle x, e_1\rangle}{\langle y, e_1\rangle} \geqslant \frac{\langle x, e_2\rangle}{\langle y, e_2\rangle} \geqslant \cdots \geqslant \frac{\langle x, e_n\rangle}{\langle y, e_n\rangle} \quad \text{(with positive denominators)}, \tag{43}$$

***(ii)*** *vectors $s_1, s_2, \ldots, s_n$ and $e_1, e_2, \ldots, e_n$ satisfy conditions (23)–(29) with matrix $M$ and vector $v$ of the form*

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 1 \end{pmatrix}$$

*and*

$$v = e_1 + e_2 + \cdots + e_n.$$

*If $n < q$, assume additionally that (35) holds.*

*Then inequality (42) holds.*

**Proof.** Employ Theorem 3.4, Lemma 2.6 and Lemma 2.5. □

## 4. Applications for weak absolute majorization

The *weak absolute majorization* (in symbols, $\prec_{wa}$) is an ordering on $\mathbb{R}^n$ defined as follows.

For vectors $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we write $y \prec_{wa} x$ if $\sum_{k=1}^{i} |y|_{[k]} \leqslant \sum_{k=1}^{i} |x|_{[k]}$ for all $i = 1, 2, \ldots, n$ (see [1, Example 2.3]).

Here we denote by $|z|_{[1]} \geqslant |z|_{[2]} \geqslant \cdots \geqslant |z|_{[n]}$ the moduli of the entries of a vector $z = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ arranged in decreasing order, where $|z| = (|z_1|, |z_2|, \ldots, |z_n|) \in \mathbb{R}^n$.

It is known that for any $x, y \in \mathbb{R}^n$,

$$y \prec_{wa} x \quad \text{iff } y \in \operatorname{conv} \mathbb{GP}_n x,$$

where $\mathbb{GP}_n$ is the group of all $n \times n$ generalized permutation matrices, and $\operatorname{conv} \mathbb{GP}_n x$ is the convex hull of the set $\mathbb{GP}_n x = \{xp : p \in \mathbb{GP}_n\}$ (see [1]).

The weak absolute majorization $\prec_{wa}$ is a GIC ordering on $\mathbb{R}^n$ induced by $\mathbb{GP}_n$ and the convex cone $\mathbb{R}^n_{+\downarrow}$ of nonincreasing nonnegative $n$-tuples (see [1, Example 2.2]). So, we have

$$V = \mathbb{R}^n \quad \text{with the standard inner product}$$

$$\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i \quad \text{for } a = (a_1, \ldots, a_n),\ b = (b_1, \ldots, b_n) \in \mathbb{R}^n, \tag{44}$$

$$G = \mathbb{GP}_n,$$
$$F = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,:\, x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0\} = \text{cone}\,\{s_1, s_2, \ldots, s_n\},$$
$$s_i = (\underbrace{1, \ldots, 1}_{i\ times}, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n,$$
$$x_\downarrow = (|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}) \quad \text{for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

By setting

$$e_i = (\underbrace{0, \ldots, 0}_{i-1\ times}, 1, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n,$$

we get

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}. \tag{45}$$

It is clear that conditions (23)–(27) are fulfilled for each $m \in \{0, 1, \ldots, n\}$.

By applying Theorem 3.4 we obtain the following.

**Corollary 4.1.** *Under the above notation, let $x, y, v \in \mathbb{R}^n$ be such that $\langle x, v\rangle > 0$, $\langle y, v\rangle > 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n > 0$.*

*Suppose that there exists $m \in \{0, 1, 2, \ldots, n\}$ such that*

$$v = e_1 + \cdots + e_m + \delta_{m+1}e_{m+1} + \cdots + \delta_n e_n$$
$$= (\underbrace{1, \ldots, 1}_{m\ times}, \delta_{m+1}, \ldots, \delta_n)$$

*for some $\delta_l \geqslant 1$, $l = m+1, m+2, \ldots, n$, and*

$$\frac{x_i}{y_i} \geqslant \frac{\langle x, v\rangle}{\langle y, v\rangle} \geqslant \frac{x_j}{y_j} \quad \text{for } i = 1, 2, \ldots, m \text{ and } j = m+1, m+2, \ldots, n.$$

*Then the following inequality holds:*

$$\frac{y}{\langle y, v\rangle} \prec_{wa} \frac{x}{\langle x, v\rangle}. \tag{46}$$

By taking $\delta_l = 1$ for $l = m+1, m+2, \ldots, n$, that is

$$v = (1, \ldots, 1) \in \mathbb{R}^n, \tag{47}$$

we obtain $v = s_n$, and then Corollary 3.5 gives an improvement of Theorem B as follows (cf. Marshall and Olkin [5, Theorem B.1.b, p. 129]).

**Corollary 4.2.** *Let $x, y, \in \mathbb{R}^n$ be such that $\sum_{k=1}^n x_k > 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n > 0$.*

*Suppose that*

$$\frac{x_1}{y_1} \geqslant \frac{x_2}{y_2} \geqslant \cdots \geqslant \frac{x_n}{y_n}.$$

*Then*

$$\frac{y}{\sum_{k=1}^{n} y_k} \prec_{wa} \frac{x}{\sum_{k=1}^{n} x_k}. \tag{48}$$

Observe that (48) easily implies the inequality

$$\frac{y}{\sum_{k=1}^{n} y_k} \prec_{wa} \frac{|x|}{\sum_{k=1}^{n} x_k}.$$

On the other hand, by setting

$$e_i = s_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n,$$

we see that conditions (23)–(27) are satisfied for the identity $n \times n$ matrix

$$M = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

By virtue of Theorem 3.4 we get the following.

**Corollary 4.3.** *Under the above notation, let $x, y, v \in \mathbb{R}^n$ be such that $\langle x, v\rangle > 0$, $\langle y, v\rangle > 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n > 0$.*

*Suppose that there exists $m \in \{0, 1, 2, \ldots, n\}$ such that*

$$\begin{aligned} v &= \delta_{m+1} e_{m+1} + \cdots + \delta_n e_n \\ &= \left( \underbrace{\sum_{k=m+1}^{n} \delta_k, \ldots, \sum_{k=m+1}^{n} \delta_k}_{m+1 \text{ times}}, \sum_{k=m+2}^{n} \delta_k, \ldots, \sum_{k=n}^{n} \delta_k \right) \end{aligned} \tag{49}$$

*for some $\delta_l \geqslant 1$, $l = m+1, m+2, \ldots, n$, and*

$$\frac{\sum_{k=1}^{i} x_k}{\sum_{k=1}^{i} y_k} \geqslant \frac{\langle x, v\rangle}{\langle y, v\rangle} \geqslant \frac{\sum_{k=1}^{j} x_k}{\sum_{k=1}^{j} y_k} \quad \text{for } i = 1, 2, \ldots, m, j = m+1, m+2, \ldots, n.$$

*Then (46) holds with vector $v$ defined by (49).*

For instance, if $\delta_l = 1$ for $l = m+1, m+2, \ldots, n$, then, in contrast to (47),

$$v = (\underbrace{n-m, \ldots, n-m}_{m+1 \text{ times}}, n-m-1, \ldots, 1) \in \mathbb{R}^n.$$

## 5. Interpretations for the classical majorization

The classical majorization $\prec_m$ is a GIC ordering on $\mathbb{R}^n$ induced by the permutation group $\mathbb{P}_n$ and the convex cone $\mathbb{R}^n_{\downarrow}$ of nonincreasing $n$-tuples (see [1, Example 2.2]). Therefore in this section we set

$$V = \mathbb{R}^n \text{ with the standard inner product given by (44)},$$

$$G = \mathbb{P}_n,$$

$$F = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n\} = \operatorname{cone}\{s_1, s_2, \ldots, s_n, s_{n+1}\},$$

$$s_i = (\underbrace{1, \ldots, 1}_{i \text{ times}}, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n, \quad \text{and } s_{n+1} = -s_n,$$

$$D = \operatorname{cone}\{s_1, s_2, \ldots, s_n\},$$

$$x_\downarrow = (x_{[1]}, x_{[2]}, \ldots, x_{[n]}) \quad \text{for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

In this situation Theorem 3.1 specializes to

**Corollary 5.1.** *Under the above notation, let $x, y, v \in \mathbb{R}^n$ with $\langle x, v\rangle\, y \in F$ (e.g., $\langle x, v\rangle \geqslant 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$).*

*Suppose that for $k = 1, 2, \ldots, n$ there exist dual bases $e = (e_1, \ldots, e_n)$ and $d = (d_1, \ldots, d_n)$ of $\mathbb{R}^n$, and index sets $I_1$ and $I_2$ with $I_1 \cup I_2 = \{1, 2, \ldots, n\}$ such that*

***(i)***

$$\frac{\langle x, e_i\rangle}{\langle y, e_i\rangle} \geqslant \frac{\langle x, v\rangle}{\langle y, v\rangle} \geqslant \frac{\langle x, e_j\rangle}{\langle y, e_j\rangle} \quad \text{for } i \in I_1 \text{ and } j \in I_2$$

*(with positive denominators),*

***(ii)***

$$\frac{\langle s_k, d_i\rangle}{\langle v, d_i\rangle} \geqslant \frac{\langle s_k, d_j\rangle}{\langle v, d_j\rangle} \quad \text{for } i \in I_1 \text{ and } j \in I_2$$

*(with positive denominators).*

*Assume additionally that*

$$\langle x, v\rangle \langle y, s_n\rangle \geqslant \langle y, v\rangle \langle x, s_n\rangle . \tag{50}$$

*Then the following majorization inequality holds:*

$$\langle x, v\rangle\, y \prec_m \langle y, v\rangle\, x. \tag{51}$$

*If $\langle x, v\rangle\ \langle y, v\rangle > 0$ then (51) becomes*

$$\frac{y}{\langle y, v\rangle} \prec_m \frac{x}{\langle x, v\rangle}. \tag{52}$$

**Remark 5.2.** Theorem A is a special case of Corollary 5.1 applied for

**(a)** vector

$$v = s_n = (1, \ldots, 1) \in \mathbb{R}^n,$$

**(b)** canonical basis in $\mathbb{R}^n$ given by

$$e_i = d_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0) \in \mathbb{R}^n \quad \text{for } i = 1, 2, \ldots, n,$$

**(c)** index sets $I_1 = \{1, 2, \ldots, m\}$ and $I_2 = \{m+1, m+2, \ldots, n\}$ for some $m \in \{1, 2, \ldots, n\}$,
**(d)** scalar $\lambda = 1$, that is $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$.

In fact, the above condition **(b)** ensures that relationship (45) holds. By Remark 2.4 and Lemma 2.5 we deduce that each vector $s_k$, $k = 1, 2, \ldots, n$, is $v$-separable w.r.t. $d$. Therefore assumption **(ii)** of Corollary 5.1 is met.

As a consequence of Theorem 3.4 we have the following.

**Corollary 5.3.** *Under the notation before Corollary 5.1, let $x, y, v \in \mathbb{R}^n$ be such that $\langle x, v\rangle > 0$, $\langle y, v\rangle > 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n$.*

*Assume that $M$ is an $n \times n$ invertible matrix of the form (25) with properties (26)–(27) and $v \in \mathbb{R}^n$ is a vector of the form (28)–(29), where*

$$\begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix} = M^{-1} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}.$$

*Suppose that there exists $m \in \{0, 1, \ldots, n\}$ such that*

$$\frac{\langle x, e_i\rangle}{\langle y, e_i\rangle} \geqslant \frac{\langle x, v\rangle}{\langle y, v\rangle} \geqslant \frac{\langle x, e_j\rangle}{\langle y, e_j\rangle} \quad \text{for } i = 1, 2, \ldots, m \quad \text{and} \quad j = m+1, m+2, \ldots, n$$

*(with positive denominators).*

*Assume additionally that (50) is satisfied.*

*Then (52) holds.*

## 6. Miranda–Thompson's majorization

Miranda–Thompson's majorization $\prec_{mt}$ is a GIC ordering on $\mathbb{R}^n$ induced by the group of generalized permutation with determinants equal to 1 (see [8, Example 2.4, 6,7]). To explain this, we consider

$V = \mathbb{R}^n$ with the standard inner product given by (44),

$G = \{g = pc \in \mathbb{P}_n\mathbb{C}_n : p \in \mathbb{P}_n, c \in \mathbb{C}_n, \det\, c = 1\}$,

where $\mathbb{P}_n$ is the group of $n \times n$ permutation matrices

and $\mathbb{C}_n$ is the group of $n \times n$ diagonal orthogonal matrices,

$$F = \{x \in \mathbb{R}^n \,:\, x_1 \geqslant x_2 \geq \cdots \geqslant x_{n-1} \geqslant |x_n|\} = \text{cone}\,\{s_1, s_2, \ldots, s_{n-2}, s_{n-1}, s_n\},$$

$$s_i = (\underbrace{1, \ldots, 1}_{i\ times}, 0, \ldots, 0) \quad \text{for } i = 1, 2, \ldots, n-2, n, \quad \text{and} \quad s_{n-1} = (1, \ldots, 1, -1),$$

$$x_\downarrow = (|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n-1]}, \mathit{sign}\,(x)\,|x|_{[n]}) \quad \text{for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n,$$

where $\text{sign}\,(x) = \text{sign}\,(\prod_{i=1}^n x_i)$ and $|x| = (|x|_1, |x|_2, \ldots, |x|_n) \in \mathbb{R}^n$.

In light of (32) and (34), for any $x, y \in \mathbb{R}^n$ we have

$$y \prec_{mt} x \quad \text{iff}$$
$$\sum_{k=1}^{i} |y|_{[k]} \leqslant \sum_{k=1}^{i} |x|_{[k]}, \quad i = 1, 2, \ldots, n-2,$$
$$\sum_{k=1}^{n-1} |y|_{[k]} - \operatorname{sign}(y)\, |y|_{[n]} \leqslant \sum_{k=1}^{n-1} |x|_{[k]} - \operatorname{sign}(x)\, |x|_{[n]},$$
$$\sum_{k=1}^{n-1} |y|_{[k]} + \operatorname{sign}(y)\, |y|_{[n]} \leqslant \sum_{k=1}^{n-1} |x|_{[k]} + \operatorname{sign}(x)\, |x|_{[n]}.$$

By employing the canonical basis

$$e_i = (\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, 1, 0, \ldots, 0) \in \mathbb{R}^n \quad \text{for } i = 1, 2, \ldots, n,$$

we infer that

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_{n-2} \\ s_{n-1} \\ s_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & 1 & 0 & 0 \\ 1 & 1 & 1 & \ldots & 1 & 1 & -1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}.$$

By setting

$$v = (1, \ldots, 1) \in \mathbb{R}^n,$$

it is easily seen that conditions (23)–(29) hold for any $m = 0, 1, \ldots, n-2$.

For this reason, Theorem 3.4 and Lemma 2.5 yield the following.

**Corollary 6.1.** *Let $x, y \in \mathbb{R}^n$ be such that $\sum_{k=1}^{n} x_k > 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{n-1} \geqslant y_n > 0$. Suppose that there exists $m \in \{0, 1, \ldots, n-2\}$ such that*

$$\frac{x_i}{y_i} \geqslant \frac{\sum_{k=1}^{n} x_k}{\sum_{k=1}^{n} y_k} \geqslant \frac{x_j}{y_j} \quad \text{for } i = 1, 2, \ldots, m, \; j = m+1, m+2, \ldots, n. \tag{53}$$

*Then the following inequality holds:*

$$\frac{y}{\sum_{k=1}^{n} y_k} \prec_{mt} \frac{x}{\sum_{k=1}^{n} x_k}.$$

**Remark 6.2.** In a more general case when

$$y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{n-1} \geqslant |y_n| \quad \text{and} \quad \sum_{k=1}^{n} y_k > 0,$$

condition (53) in Corollary 6.1 should be replaced by

$$x_i \geqslant y_i \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n y_k} \quad \text{and} \quad y_j \frac{\sum_{k=1}^n x_k}{\sum_{k=1}^n y_k} \geqslant x_j \quad \text{for } i = 1, 2, \ldots, m, \; j = m+1, m+2, \ldots, n.$$

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